THE FIRST ALGEBRA BOOK IN THE OTTOMAN EMPIRE: AN ANALYSIS OF KHAYR AL-DĪN KHALĪL IBN IBRĀHĪM'S *MUSHKILKUSHĀ-YI ḤUSSĀB WA MUʿḌILNUMĀ-YI KUTTĀB*

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Abstract

Khayr al-dīn Khalīl ibn Ibrāhīm was a 15th-century Ottoman mathematician known for his works in Persian. Some sources suggest he might be the same as Khodja Khayr al-dīn, the tutor of Sultan Mehmed II, though this remains uncertain. His most notable works are *Miftāḥ-i Kunūz-i Erbāb al-Kalam wa Misbāḥ-i Rumūz-i Ashāb al-Raḳam* and *Mushkilkushā-yı Ḥussāb wa Muʿḍılnumā-yı Kuttāb*, the latter being an advanced algebra book dedicated to Sultan Bayezid II.

Miftāḥ-i Kunūz was a foundational arithmetic book used by Ottoman accountants, covering topics such as fractions, divisibility, proportional numbers, multiplication, division,

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debt calculation, and root extraction. It was later translated into Turkish by his student Pīr Maḥmud Ṣıdkī Edirnevī.

Mushkilkushā-yı Ḥussāb is a six-chapter algebra book that includes definitions, methods, and advanced problem-solving techniques. It covers quadratic, cubic, and higher-degree equations, including unique methods not commonly found in classical algebra texts. The manuscript in the Ayasofya Collection suggests it was copied for a royal audience.

This study aims to analyze the introduction and first chapter of *Mushkilkushā-yı Ḥussāb* by examining its manuscripts from the Ayasofya and Şehid Ali Paşa collections. The research explores the mathematical content, methodology, and instructional approach of the book, highlighting its significance as the earliest known algebra text written in the Ottoman Empire. By investigating the problem-solving techniques presented in the work, this study sheds light on the advanced mathematical training required for Ottoman accountants and scribes.

Keywords: History of science, Khayr al-dīn Khalīl ibn Ibrāhīm, *Mushkilkushā-yı Ḥussāb,* history of mathematics in the Ottoman Empire, algebra

1. KHAYR AL-DĪN KHALĪL IBN İBRĀHĪM AND HIS MATHEMATICAL WORKS

Khayr al-dīn Khalīl ibn Ibrāhīm (9th century AH / 15th century CE) was a mathematician from the period of Mehmed II the Conqueror, who wrote his works in Persian. Although little is known about his life, some sources, referencing Tashkopruluzāde's work *al-Shaqāik al-Nuʿmāniyya*, suggest that he might be the same person as Khodja Khayr al-dīn, the tutor of Sultan Mehmed II the Conqueror. According to Tashkopruluzāde, Khodja Khayr al-dīn was educated by Hızır Bey, taught at various madrasas in Istanbul, and later became the Sultan's tutor. The same source states that he passed away towards the end of Sultan Mehmed's reign.[1] However, since Khodja Khayr al-dīn Khalīl dedicated his works *Mushkilkushā-yı Ḥussāb and Mu'ḍılnumā-yı Kuttāb* to Sultan Bayezid II, either the information regarding his death is incorrect, or Khayr al-dīn Khalīl and Khodja Khayr al-dīn were different individuals.

Khayr al-dīn Khalīl ibn Ibrāhīm is known for two mathematical works written in Persian. He is particularly recognized for his first work, *Miftāḥ-i Kunūz-i Erbāb al-Kalam wa Misbāḥ-i Rumūz-i Ashāb al-Raḳam*, a treatise on accounting presented to Sultan Mehmed II. This work became renowned in the Ottoman realm as it provided fundamental arithmetic knowledge required by accountants. It was later translated into Turkish by his student Pīr Mahmud Sıdkī Edirnevī.[2]

The treatise consists of an introduction explaining basic definitions and methods, followed by ten chapters on various arithmetic topics, and concludes with examples in the final section. The introduction elucidates units of measurement used in commercial

1 İhsan Fazlıoğlu, "Osmanlı Klasik Muhasebe Matematik Eserleri Üzerine Bir Değerlendirme," *Türkiye Araştırmaları Literatür Dergisi* 1, no. 1 (2003): 351; Ekmeleddin İhsanoğlu vd., *Osmanlı Matematik Literatürü Tarihi*, Volume 1 (IRCICA, 1999), 33.

2 İhsanoğlu vd., *Osmanlı Matematik Literatürü Tarihi*, 34; Tuba Oguz Ceyhan, "Sayısal Analiz Metotlarının Kısa Tarihi ve Bu Bağlamda Pîr Mahmud Sıdkı Edirnevî'nin Hesap Kitabı," *Turkish Journal of History* 84 (2024): 1-34.

arithmetic and their conversions. Subsequent sections discuss fractions, divisibility rules, arithmetic operations with fractions, and proportional numbers.

The first chapter covers multiplication of fractions, explaining how to multiply various combinations of fractions, whole numbers, and mixed numbers. The following three chapters elaborate on the multiplication of the units defined earlier. The fifth chapter explores division, similar to the multiplication section, but focusing on the division of various types of fractions and integers. The sixth chapter addresses debt calculation, while the seventh examines the double false position method. Chapters eight through ten detail the methods for extracting square roots, cube roots, and fourth roots, respectively.[3]

Besides *Miftāḥ-i Kunūz*, Khayr al-dīn Khalīl ibn Ibrāhīm also authored another Persian treatise, *Mushkilkushā-yı Ḥussāb and Mu'ḍılnumā-yı Kuttāb*, an advanced algebra book dedicated to Sultan Bayezid II. Manuscripts of this work are in the Süleymaniye Library's Ayasofya Collection (no. 2731) and Şehid Ali Paşa Collection (no. 1993).[4] The Ayasofya manuscript is particularly noteworthy for its regal presentation, with the inscription "Shahzāde Sultan Mehmed" on the folio page, indicating that it was likely copied for a prince.

Mushkilkushā-yı Ḥussāb consists of a preface, an introduction, six chapters, and a conclusion. In the preface, the author narrates the reason for writing his work, stating that while he was in seclusion, his friends came to him and requested him to compose a few chapters on the science of calculation. They asked him to write an exceptional treatise that, in addition to the commonly known knowledge, would also include subjects unique to the author himself. Khayr al-dīn Khalīl highlights the distinction of his work from existing studies by stating, *"the scholars of old and the meticulous philosophers had not arranged, refined, adorned, and structured this science and knowledge in such a manner."*

The introduction provides definitions necessary for algebraic operations, such as powers of algebraic quantities, multiplication, positive and negative expressions, completion (*tekmīl*), reduction

3 Hayruddin Halil b. İbrahim, *Miftâh-i Kunûz-i Erbâbi'l-Kalem ve Misbâh-i Rumûz-i Ashâbi'r-rakam*, (Süleymaniye Library, Şehid Ali Paşa, 1973/2) 31b-78a.

4 Hayruddin Halil b. İbrahim, *Müşkilkuşâ-yı Hussâb ve Mu'zılnumâ-yı Kuttab* (Süleymaniye Library, Ayasofya, 2731) 1b-76b; (Şehid Ali Paşa, 1993) 1b-65b.

(*redd*), balancing (*muqābala*), equality (*mu'adala*), and mensuration (*mesāha*). The second chapter includes various examples that will be solved using six equations. The third chapter consists of a range of problems. The fourth chapter discusses the solutions of cubic equations, while the fifth chapter examines equations of the fourth, fifth, and sixth degrees. The sixth chapter is dedicated to mensuration, exploring the fundamental definitions and properties of geometry. In the conclusion, surface and volume problems are addressed.

The discussion of solutions for cubic and higher-degree equations, as well as methods not commonly found in standard algebra or calculus textbooks—such as solving equations without reduction or completion—indicates that the work is intended primarily for advanced algebra education rather than for general readers.

Apart from the two works mentioned in the sources, a Persian algebra treatise recorded in Manuscript No. 2980 of the Nuruosmaniye Collection at the Süleymaniye Library is attributed to Khayr al-dīn Khalīl ibn Ibrāhīm. However, as there is no colophon indicating its belonging to Khayr al-dīn Khalīl, the phrase written in the title—"رساله جبر ومقابله لبنی خیرالدین" (The Algebra Treatise of Khayr al-dīn's Son)—suggests that the work actually belongs to his son rather than to himself. This treatise represents a variation of the introduction to *Mushkilkushā-yı Ḥussāb*. The author's son included the theoretical content from his father's work but transcribed only a limited selection of the examples.

The most intriguing part of the treatise appears at the end, where a poetic mathematical problem is presented. This problem, which has been frequently encountered in Persian and Turkish versions of accounting mathematics books since the 15th century CE, is—based on current knowledge—first found in Khayr al-dīn Khalīl ibn Ibrāhīm's renowned work *Miftāḥ-i Kunūz*.[5]

The problem describes a merchant who possesses three precious stones: a diamond, a ruby, and a garnet. The prices of these stones are expressed in relation to a fixed amount of 1,000 Florins, from which specific fractions of the prices of the other stones are deducted. According to the problem, if one-third of the diamond's price is subtracted from 1,000 Florins, the result is the ruby's price.

5 For fifteenth and sixteenth century sources containing this verse problem, see Atilla Polat, "15. ve 16. Yüzyıl Türkçe Matematik Eserlerinde Geçen Manzum Bir Matematik Problemi," *Osmanlı Bilimi Araştırmaları* 22 no. 2 (2021): 241-253.

If half of the ruby's price is subtracted from 1,000 Florins, the result is the price of the garnet. If one-fourth of the garnet's price is subtracted from 1,000 Florins, the result is the price of the diamond. The sum of the prices of all three stones is given as 2,200 Florins.

When representing the prices of the diamond, ruby, and garnet as x, y, and z, respectively, the given information provides the following system of equations:

$$y = 1000 - \frac{x}{3}$$

$$z = 1000 - \frac{y}{2}$$

$$x = 1000 - \frac{z}{4}$$

$$x + y + z = 2200$$

To solve the problem, the price of the ruby (y) is taken as the base variable, and the prices of the other stones are expressed in terms of y:

$$x = 1000 - \frac{1000 - \frac{y}{2}}{4} = 750 + \frac{y}{8}$$

$$x + y + z = 750 + \frac{y}{8} + y + 1000 - \frac{y}{2}$$

$$2200 = 1750 + \frac{5y}{8}$$

Applying the method of *muqābala* (balancing), we have

$$450 = \frac{5y}{8}$$

and

$$y = 720$$

Thus, the price of the ruby is found to be 720. Using this value to find the prices of the other stones: $z = 640$ and $x = 840$.

What is particularly interesting here is that the problem, which Hayreddin Halil b. İbrahim originally solved using the *double false position*[6] method in *Miftāḥ-i Kunūz*, was instead solved algebraically by his son in this treatise. In all previously identified Turkish

6 For double false notation in *Miftāḥ-i Kunūz,* see Tuba Oguz Ceyhan, "Klasik Dönem Osmanlı Matematiğinde Pîr Mahmud Sıdkı Edirnevî'nin "Çift Yanlış" Metodu" *Erdem* 79, (2020): 149-174.

and Persian works containing this problem, the *double false position* method had been consistently applied. The fact that Khalīl ibn İbrāhīm's son opted for an algebraic approach, which offers a more straightforward solution, is noteworthy.

Another indication of the widespread influence of *Mushkilkushā-yı Ḥussāb* is the remarkable similarity between the algebra section of *Turkī Ḥisāb*,[7] the earliest known Turkish mathematics book of unknown authorship, and the introduction of *Mushkilkushā-yı Ḥussāb* —suggesting that the former might be an almost direct translation of the latter. Furthermore, the sections on Indian arithmetic and the *double false position* method in *Turkī Ḥisāb* closely resemble those found in *Miftāḥ-i Kunūz*. These similarities suggest that the author of *Turkī Ḥisāb* may have compiled his work by translating and merging various sections from two of Khayr al-dīn Khalīl ibn İbrāhīm's books—one focused on arithmetic and the other on algebra. However, considering the copying date of *Turkī Hisāb* in the year 1461 CE and the fact that Khayr al-dīn Khalīl ibn İbrāhīm dedicated *Mushkilkushā-yı Ḥussāb* to Sultan Bayezid II, it becomes evident that further evidence is needed to resolve the chronological discrepancy.

This study presents a mathematical interpretation of the introduction and first chapter of *Mushkilkushā-yı Ḥussāb*, using the manuscripts from Ayasofya and Şehid Ali Paşa collections as primary sources. This analysis offers insights into the content and methodology of the earliest algebra book written in the Ottoman realm, demonstrating the rigorous training in algebra required of accountants at the time.

2. MATHEMATICAL ANALYSIS

2.1. Introduction

The article begins with nine definitions.

In the first section, the quantities used in algebraic operations are introduced. Initially, the acquisition and naming of unknowns and their powers are explained. According to this:

7 For a translation of the work, see Şermin Kalafat, E*ski Anadolu Türkçesiyle Yazılmış Bir Matematik Kitabı Türkî Hisāb: İzleksel Terimbilimsel İnceleme* (Kesit Publishing, 2019).

i. *Square* (*māl, murabba‘, meczūr*): The product of a number multiplied by itself.

ii. *Root* (cezr, shey): The number being multiplied by itself to obtain a square.

iii. *Cube* (*ka‘b, muḳa‘ab*): Obtained by multiplying the root by its square.

iv. *Square-square* (*māl-i māl*): The product of the cube and the root.

v. *Square-cube* (*māl-i ka‘b*): Obtained by multiplying the square-square by the root.

vi. *Cube-cube* (*ka‘b-i ka‘b*): The product of square-cube and the root.

vii. *Square-square-cube* (*māl-i māl-i ka‘b*): Obtained by multiplying the cube-cube by the root.

viii. *Square-cube-cube* (*māl-i ka‘b-ı ka‘b*): The product of square-square-cube and the root.

ix. *Cube-cube-cube* (*ka‘b-ı ka‘b-ı ka‘b*): Obtained by multiplying square-cube-cube by the root.

Higher powers are obtained by arranging *māl* and *ka‘b* together.

Given a number, the above definitions are expressed as:

i. Square: x^2

ii. Root: $x = \sqrt{x^2}$

iii. Cube: x^3

iv. Square-square: x^4

v. Square-cube: x^5

vi. Cube-cube: x^6

vii. Square-square-cube: x^7

viii. Square-cube-cube: x^8

ix. Cube-cube-cube: x^9

The relationships between these powers can be observed in the sequence. The ratio of one power to the next is constant, indicating a geometric progression.

For any number x different from zero,

$$\frac{1}{x} = \frac{x}{x^2} = \frac{x^2}{x^3} = \frac{x^3}{x^4} = \frac{x^4}{x^5} = \frac{x^5}{x^6} = \frac{x^6}{x^7} = \frac{x^7}{x^8} = \frac{x^8}{x^9} = \cdots$$

and

$$\frac{x^3}{x} = \frac{x^2}{1}$$

and similar ratios can be obtained.

Similarly, a proportional relationship exists among the fractional components of the root. The ratio of unity to the root is defined as the *fraction of the root* (*cuz'-i cezr*). The number whose ratio to the *fraction of the root* is equal to the ratio of the *fraction of the root* to unity is called the *fraction of the square* (*cuz'-i māl*). The number whose ratio to the *fraction of the square* is equal to the ratio of the *fraction of the square* to the *fraction of the root* is termed the *fraction of the cube* (*cuz'-i ka'b*). Likewise, the number whose ratio to the *fraction of the cube* is equal to the ratio of the *fraction of the cube* to the *fraction of the square* is called the *fraction of the square-square* (*cuz'-i māl-i māl*).

According to these definitions, for any nonzero number x:

i. The *fraction of the root* is $\frac{1}{x}$
ii. The *fraction of the square* is $\frac{1}{x^2}$.
iii. The *fraction of the cube* is $\frac{1}{x^3}$.
iv. The *fraction of the square-square* is $\frac{1}{x^4}$.

This pattern can be extended further to higher fractional powers. Additionally, a proportional relationship exists among the terms of this sequence.

$$\frac{1}{x} = \frac{\frac{1}{x^2}}{\frac{1}{x}} = \frac{\frac{1}{x^3}}{\frac{1}{x^2}} = \frac{\frac{1}{x^4}}{\frac{1}{x^3}} = \cdots$$

After introducing algebraic terms in the treatise, the second section transitions to multiplication and division.

Multiplication is divided into two categories: simple multiplication and compound multiplication.

a. In simple multiplication, both the multiplicand and the multiplier are single terms, and their coefficients are equal to one. Examples include:
   i. *Root* multiplied by *root*
   ii. *Square* multiplied by *square*
   iii. *Cube* multiplied by *cube*
   iv. *Root* multiplied by *square*
   v. *Square* multiplied by *root*

This classification establishes a structured approach to multiplication within algebraic operations.

Let x be an algebraic quantity. Then the multiplications

$$x \cdot x\,,\;\; x^2 \cdot x^2\,,\;\; x^3 \cdot x^3\,,\;\; x \cdot x^2\,,\;\; x^2 \cdot x$$

are simple multiplications.

b. In compound multiplication, both the multiplicand and the multiplier consist of multiple terms and may have different coefficients. Examples include:
   i. (Root and square) multiplied by (a number and a root)
   ii. (A number and a root) multiplied by (a square and a cube)
   iii. (A number and a root) multiplied by (a square-cube and a square-square)

This type of multiplication extends beyond simple monomials, involving more complex algebraic expressions and their interactions.

Let a be a number and x be an algebraic quantity. Then

$$(x + x^2) \cdot (a + x), \qquad (a + x) \cdot (x^2 + x^3), \qquad (a + x) \cdot (x^5 + x^4)$$

are compound multiplications.

Performing simple multiplication is straightforward. The exponents of the multiplied variables are added together and written as the exponent of the result.

$$x \cdot x = x^2\,, \quad x^2 \cdot x^2 = x^4\,, \quad x^3 \cdot x^3 = x^6, \quad x^3 \cdot x^2 = x^5\,, \quad x^2 \cdot x = x^3$$

In compound multiplication, multiple simple multiplications must be performed.

For example, if we want to multiply two roots and one square and five by two roots and one square and five, we need to apply the distributive property and perform nine separate multiplications, as each term in the first expression must be multiplied by each term in the second expression.

$$(2x + x^2 + 5) \cdot (2x + x^2 + 5)$$
$$= 2x \cdot 2x + x^2 \cdot x^2 + 5 \cdot 5 + 2x \cdot x^2 + x^2 \cdot 2x + 2x \cdot 5 + 5 \cdot 2x + x^2 \cdot 5 + 5 \cdot x^2$$
$$= 25 + 20x + 14x^2 + 4x^3 + x^4$$

Division is the inverse of multiplication, so dividing monomials is relatively simple. When variables of the same degree are divided, the result is a numerical quotient.

If the degree of the dividend (the numerator) is the same as the degree of the divisor (the denominator), the quotient is expressed as a number.

However, if the degree of the dividend differs from the degree of the divisor, the quotient will be a term that, when multiplied by the divisor, gives back the dividend. For example,

$$ax^2 \div bx = \frac{a}{b}x,$$

$$ax \div b = \frac{a}{b}x,$$

$$ax \div bx = \frac{a}{b}$$

A monomial cannot be divided by a binomial or polynomial. However, the reverse is possible. For example, if we divide ten cubed and ten squared minus ten root minus ten by two roots, four operations are needed to solve this problem.

$$\begin{aligned}(10x^3 + 10x^2 - 10x - 10) \div 2x \\ &= (10x^3 \div 2x) + (10x^2 \div 2x) + (-10x \div 2x) + (-10 \div 2x) \\ &= 5x^2 + 5x - 5 - \frac{5}{x}\end{aligned}$$

In the third section, the multiplication of expressions with positive (*musbet*) and negative (*menfī*) terms is examined. For example, in the expression two squares minus two roots, two squares is positive, and two roots is negative. The rules for multiplying positive and negative terms are provided. According to these rules:

i. The product of positive and positive terms is positive:
$(+) \cdot (+) = +$

ii. The product of negative and negative terms is also positive:
$(-) \cdot (-) = +$

iii. The product of positive and negative terms is negative:
$(+) \cdot (-) = -$

For example,
$(x^2 \cdot x^2) + (-2x \cdot -2x) + (2x^2 \cdot -2x) - 4x^3 = x^4 + 4x^2 - 8x^3$

In the fourth section, the process of completion (*tekmīl*) is discussed. Completion refers to the process of expanding an equation so that terms with coefficients smaller than 1 are adjusted to have a coefficient of 1. This is done by adding a term that transforms the fractional term into a whole term and increasing the other terms proportionally to maintain the equation's balance. For example,

$$\frac{3}{8}x^3 + 15x^2 = 60x + 24$$

To complete the term $\frac{3}{8}x^3$ to x^3, we need to add $\frac{5}{8}x^3$. This operation corresponds to increasing $\frac{3}{8}x^3$ by a factor of $\frac{5}{3}$. When this increase is applied to all terms in the equation, the result is:

$$\frac{3}{8}x^3 + \frac{5}{3}\cdot\frac{3}{8}x^3 + 15x^2 + \frac{5}{3}\cdot 15x^2 = 60x + \frac{5}{3}\cdot 60x + 24 + \frac{5}{3}\cdot 24$$

$$x^3 + 40x^2 = 160x + 64$$

In the fifth section, algebra operation (*al-jabr*) are explained. The algebra operation involves adding a term with negative sign in one side of the equation to both sides of the equation in order to complete the missing part. For example,

$$7x - 5 = 23$$

$$7x - 5 + 5 = 23 + 5$$

$$7x = 28$$

$$x = 4$$

In the sixth section, reduction (*redd*) operations are examined. This operation is the reverse of the completion operation, meaning it involves reducing the equation so that terms with coefficients greater than one are adjusted to have a coefficient of one.

The reduction process is carried out by subtracting a value from the term with a coefficient greater than 1 in such a way that its coefficient becomes 1. Other terms are then simplified using the same ratio. For example,

$$3x^3 + 3x^2 = 60x$$

To reduce the term $3x^3$ to x^3, we need to subtract $2x^3$. This operation reduces $3x^3$ by a factor of $\frac{2}{3}$. When this ratio is applied to all other terms in the equation, the following equation is obtained:

$$3x^3 - \frac{2}{3}\cdot 3x^3 + 3x^2 - \frac{2}{3}\cdot 3x^2 = 60x - \frac{2}{3}\cdot 60x$$

$$x^3 + x^2 = 20x$$

In the seventh section, the balancing operation (*muqābala*) is discussed. The balancing operation refers to the simplification of terms of the same degree on both sides of an equation. In this case, the smaller term is subtracted from both sides of the equation. For example,

$2x^2 + 4x = 10x$

$2x^2 + 4x - 4x = 10x - 4x$

$2x^2 = 6x$

Another example:

$2x^3 + 2x^2 + 5x - 2 = x^3 + x^2 + 17x - 2$

$2x^3 - x^3 + 2x^2 + 5x - 2 = x^3 - x^3 + x^2 + 17x - 2$

$x^3 + 2x^2 + 5x - 2 = x^2 + 17x - 2$

$x^3 + 2x^2 - x^2 + 5x - 2 = x^2 - x^2 + 17x - 2$

$x^3 + x^2 + 5x - 2 = 17x - 2$

$x^3 + x^2 + 5x - 5x - 2 = 17x - 5x - 2$

$x^3 + x^2 - 2 = 12x - 2$

$x^3 + x^2 - 2 + 2 = 12x - 2 + 2$

$x^3 + x^2 = 12x$

In the eighth section, the concept of equality is mentioned. Equality refers to the state where two expressions are balanced or equivalent in value.

In the ninth and final section, *mesāha* (mensuration) is defined. *Mesāha* refers to the methods of calculating areas and volumes, which are crucial for solving geometric problems involving space and surfaces.

After these definitions, the main body of the text, which explains the theory of equations, follows. This section includes the various types of equations and the methods for solving them, providing a comprehensive overview of algebraic principles related to equations.

2.2. The six equation forms

The six equation forms of algebra are introduced in the first chapter. These six forms are categorized into two types: three discrete (*mufterikat*) forms and three compound (*mukterinat*) forms. The first three are called discrete since there is only one term on each side of the equation. Conversely, the other three are called compound as there is more than one term on one side of the equation.

The discrete equations are as follows:

i. $\boldsymbol{bx = c}$

Examples:

a. If $29 = 5x$, then what is x?

$$\frac{29}{5} = x$$

$$x = 5\frac{4}{5}$$

b. If $\frac{1}{7}x = 5\frac{1}{3}$, then what is x?

$$6 \cdot \frac{1}{7}x + \frac{1}{7}x = 6 \cdot \left(5\frac{1}{3}\right) + 5\frac{1}{3}$$

$$x = 37\frac{1}{3}$$

c. If $\frac{1}{9}x = \frac{3}{4} + \frac{1}{5}$, then what is x?[8]

$$\frac{1}{9}x = \frac{20}{20}\left(\frac{3}{4} + \frac{1}{5}\right)$$

$$\frac{1}{9}x = \frac{19}{20}$$

$$x = \frac{19 \times 9}{20} = \frac{171}{20} = 8 + \frac{1}{2} + \frac{1}{10} \cdot \frac{1}{2}$$

ii. $\boldsymbol{ax^2 = bx}$

Examples:

a. If $11x = 3x^2$, then what is x?

$$x = \frac{11}{3} = 3\frac{2}{3}$$

$$x^2 = 13\frac{4}{9}$$

b. If $2x^2 + \frac{1}{7}x^2 = 10x + \frac{5}{7}x$, then what is x?

$$\frac{14}{7}x^2 + \frac{1}{7}x^2 = \frac{70}{7}x + \frac{5}{7}x$$

$$\frac{15}{7}x^2 = \frac{75}{7}x$$

8 Here, $\frac{3}{4} + \frac{1}{5}$ is considered as one number written in basic fractions (fractions with numerator 1, and the fraction $\frac{2}{3}$), rather than the sum of two separate fractions.

$$x = \frac{\frac{75}{7}}{\frac{15}{7}} = 5$$

$$x^2 = 25$$

c. If $\frac{1}{5}x^2 = 3x$, then what is x?

$$\frac{1}{5}x^2 + \frac{4}{5}x^2 = 3x + 4 \cdot 3x$$

$$x^2 = 15x$$

$$x = 15$$

$$x^2 = 225$$

iii. $\boldsymbol{ax^2 = c}$

Examples:

a. If $3x^2 + \frac{x^2}{3} = 30$, then what is x^2?

$$\frac{10}{3}x^2 = 30$$

$$10x^2 = 90$$

$$x^2 = 9$$

The problems in these three discrete forms are all solved using similar methods.

Afterward, we move on to compound equations. In compound equations, if the coefficient of x^2 is not 1, the equation is transformed to have a coefficient of 1 through either the reduction or completion process.

The compound equations are as follows:

i. $\boldsymbol{ax^2 + bx = c}$

To solve this form of the equation, first, the coefficient of x^2 is transformed to 1 if it is greater or less than 1. Then, the following steps are taken:

1. Half of the coefficient of x is squared and added to the constant term of the equation.
2. The square root of the resulting value is taken.

3. Half of the coefficient of x is subtracted from the square root.
4. The resulting value is the root of the equation.

That is, for $x^2 + bx = c$,

$$x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}$$

We note that here, as well as in the following two forms, only positive roots are found.

Example:

a. If $5x + x^2 = 26 + \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4}$, then what is x, what is x^2?

$$\left(2\frac{1}{2}\right)^2 + 26 + \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} = 6\frac{1}{4} + 26 + \frac{3}{4} + \frac{1}{16} = 33\frac{1}{16}$$

$$\sqrt{33\frac{1}{16}} = 5\frac{3}{4}$$

$$5\frac{3}{4} - 2\frac{1}{2} = 3\frac{1}{4}$$

$$x = 3\frac{1}{4}$$

$$x^2 = 10 + \frac{1}{2} + \frac{1}{16}$$

If the coefficient of the square term is greater than 1, it needs to be reduced to 1. If the coefficient is smaller than 1, it needs to be rounded up to 1.

To do this, if the coefficient is greater than 1, excess squares are subtracted to reduce the coefficient to one. If the coefficient is smaller than 1, missing squares are added to complete the term to 1.

Additionally, the same amount of reduction or completion is applied to the other terms in the equation to maintain balance.

Example:

b. If $4x^2 + 12x = 55$, then what is x ?

$$4x^2 + 12x = 55$$

$$x^2 + 3x = 13\frac{3}{4}$$

$$\left(1\frac{1}{2}\right)^2 + 13\frac{3}{4} = 16$$

$$\sqrt{16} = 4$$

$$4 - 1\frac{1}{2} = 2\frac{1}{2}$$

$$x = 2\frac{1}{2}$$

$$x^2 = 6\frac{1}{4}$$

c. If $\frac{2}{5}x^2 + 11x = 30$, then what is x, what is x^2?

$$\frac{2}{5}x^2 + 11x = 30$$

$$x^2 + 27\frac{1}{2}x = 75$$

$$\left(\frac{27\frac{1}{2}}{2}\right)^2 + 75 = 264\frac{1}{16}$$

$$\sqrt{264\frac{1}{16}} = 16\frac{1}{4}$$

$$16\frac{1}{4} - 13\frac{3}{4} = 2\frac{1}{2}$$

$$x = 2\frac{1}{2}$$

If the coefficient of x^2 is either greater than 1 or smaller than 1, but does not want to be changed, the following steps are followed:

1. Half of the coefficient of x is squared and added to the product of the constant number and the coefficient of x^2.
2. The square root of the resulting value is taken.
3. Half of the coefficient of x is subtracted from the square root.
4. The resulting value is divided by the coefficient of x^2.
5. The result is the root of the equation.

That is, for the equation $ax^2 + bx = c$,

$$x = \frac{\sqrt{\left(\frac{b}{2}\right)^2 + ac} - \frac{b}{2}}{a}$$

Examples:

d. If $2x^2 + 5x = 33$, then what is x, what is x^2?

$$2x^2 + 5x = 33$$

$$\left(2\frac{1}{2}\right)^2 + 2 \cdot 33 = 6\frac{1}{4} + 66 = 72\frac{1}{4}$$

$$\sqrt{72\frac{1}{4}} = 8\frac{1}{2}$$

$$8\frac{1}{2} - 2\frac{1}{2} = 6$$

$$\frac{6}{2} = 3$$

$$x = 3$$

$$x^2 = 9$$

e. If $\frac{2}{5}x^2 + 11x = 30$, then what is x, what is x^2?

$$\left(5\frac{1}{2}\right)^2 + \frac{2}{5} \cdot 30 = 30\frac{1}{4} + 12 = 42\frac{1}{4}$$

$$\sqrt{42\frac{1}{4}} = 6\frac{1}{2}$$

$$6\frac{1}{2} - 5\frac{1}{2} = 1$$

$$\frac{1}{\frac{2}{5}} = \frac{5}{2} = 2\frac{1}{2}$$

$$x = 2\frac{1}{2}$$

$$x^2 = 6\frac{1}{4}$$

i. $\boldsymbol{ax^2 + c = bx}$

In solving this equation form, the coefficient of x^2 is first transformed to 1 if it is greater or less than 1. Then, Half of the coefficient of x is squared, and the constant term on the side of x^2 is subtracted from this square.

1. If the square of half the coefficient of x is smaller than the constant term, the equation has no solution.
2. Otherwise, the square root of the resulting value is taken.
3. The square root is subtracted from half the coefficient of x.
4. The final result is equal to x.

That is, for $x^2 + c = bx$ and $\left(\frac{b}{2}\right) \geq c$,

$$x = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

Example:

a. If $3x^2 + 48 = 25x$, then what is x, what is x^2?

$$x^2 + 16 = 8x + \frac{1}{3}x$$

$$\left(\frac{8\frac{1}{3}}{2}\right)^2 - 16 = 1 + \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{6}$$

$$\sqrt{1 + \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{6}} = 1\frac{1}{6}$$

$$\frac{8\frac{1}{3}}{2} - 1\frac{1}{6} = 3$$

$$x_1 = 3 \ , \qquad x_1^2 = 9$$

In this case, the equation has another positive root, and it can be found by subtracting the first root from b.

$$8\frac{1}{3} - 3 = 5\frac{1}{3}$$

$$x_2 = 5\frac{1}{3} \ , \qquad x_2^2 = 28 + \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = 28 + \frac{4}{9}$$

If the square of half the coefficient of x is equal to the constant term, that is,

$$\left(\frac{b}{2}\right)^2 = c$$

the roots are equal to half the coefficient of x.

Example:

b. If $x^2 + 9 = 6x$, then what is x, what is x^2?

$$\left(\frac{6}{2}\right)^2 = 9$$

Since the constant is also 9

$$x = 3\ ,\ \ x^2 = 9$$

If a solution is sought for the equation $ax^2 + c = bx$ when the coefficient of x^2 is either greater or less than one, the following value is calculated:

$$\frac{\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - cb}}{a}$$

These values represent the roots of the equation.

Example:

c. If $2x^2 + 40 = 18x$, then what is x, what is x^2?

$$\left(\frac{18}{2}\right)^2 = 81$$

$$40 \cdot 2 = 80$$

$$81 - 80 = 1$$

$$\sqrt{1} = 1$$

$$\frac{18}{2} - 1 = 8$$

$$x_1 = \frac{8}{2} = 4, \qquad x_1^2 = 16$$

$$\frac{18}{2} + 1 = 10$$

$$x_2 = \frac{10}{2} = 5, \qquad x_2^2 = 25$$

If x^2 is to be determined first in the equation $ax^2 + c = bx$, the following method is applied:

1. The square of the coefficient of x is taken and multiplied by the constant term. This result is retained.
2. Separately, the square of half the square of the coefficient of x is computed. The retained result is subtracted from this value.
3. The square root of the remainder is taken.

4. Adding this root to half the square of the coefficient of x and then subtracting the constant term yields one possible value of x^2.
5. Conversely, subtracting the obtained root from half the square of the coefficient of x and then subtracting the constant term yields the other possible value of x^2.

In summary, for the equation $x^2 + c = bx$, the values of x^2 are given by:

$$x_1^2 = \frac{b^2}{2} + \left(\sqrt{\left(\frac{b^2}{2}\right)^2 - b^2c}\right) - c$$

$$x_2^2 = \frac{b^2}{2} - \left(\sqrt{\left(\frac{b^2}{2}\right)^2 - b^2c}\right) - c$$

Example:

d. If $x^2 + 18 = 9x$, then what is x^2?

$$9^2 = 81$$

$$81 \cdot 18 = 1458$$

$$\left(\frac{9^2}{2}\right)^2 = 1640\frac{1}{4}$$

$$1640\frac{1}{4} - 1458 = 182\frac{1}{4}$$

$$\sqrt{182\frac{1}{4}} = 13\frac{1}{2}$$

$$x_1^2 = \frac{81}{2} + 13\frac{1}{2} - 18 = 54 - 18 = 36$$

$$x_1 = 6$$

$$x_2^2 = \frac{81}{2} - 13\frac{1}{2} - 18 = 27 - 18 = 9$$

$$x_2 = 3$$

iii. $\boldsymbol{bx + c = ax^2}$

First, the coefficient of the square term is adjusted to 1 using the reduction or completion method. After this, the equation is solved by computing the following value:

$$\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}$$

This value represents the root of the equation.

Examples:

a. If $3x + 4 = x^2$, then what is x, what is x^2?

$$\left(\frac{3}{2}\right)^2 + 4 = 6 + \frac{1}{4}$$

$$\sqrt{6 + \frac{1}{4}} = 2 + \frac{1}{2}$$

$$2 + \frac{1}{2} + \frac{3}{2} = 4$$

$$x = 4, \ \ x^2 = 16$$

b. If $2x^2 = 5x + 12$, then what is x, what is x^2?

$$x^2 = 2\frac{1}{2}x + 6$$

$$\left(\frac{2\frac{1}{2}}{2}\right)^2 + 6 = 7 + \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4}$$

$$\sqrt{7 + \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4}} + \frac{2\frac{1}{2}}{2} = 4$$

$$x = 4, \ \ x^2 = 16$$

c. If $\frac{3}{7}x^2 = 2x + 7$, then what is x, what is x^2?

First, the coefficient of x^2 is completed to 1, and the same completion is applied to other terms as well.

$$\frac{3}{7}x^2 + \frac{4}{7}x^2 = 2x + 2\frac{2}{3}x + 7 + 9\frac{1}{3}$$

$$x^2 = 4\frac{2}{3}x + 16\frac{1}{3}$$

$$\left(\frac{4\frac{2}{3}}{2}\right)^2 + 16\frac{1}{3} = 21\frac{7}{9}$$

$$\sqrt{21\frac{7}{9}} + \frac{4\frac{2}{3}}{2} = 4\frac{2}{3} + 2\frac{1}{3} = 7$$

$$x = 7, \quad x^2 = 49$$

If the equation is to be solved without reducing or completing the coefficient of x^2 to 1, the following method is applied:

1. The square of half the coefficient of x is added to the product of the coefficient of $x^2$and the constant term c.
2. The square root of this sum is taken.
3. This root is added to half the coefficient of x.
4. The result is then divided by the coefficient of x^2.

In summary, the solution to the equation is given by:

$$x = \frac{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + ac}}{a}$$

Example:

a. If $4x^2 = 12x + 16$, then what is x, what is x^2?

$$\left(\frac{12}{2}\right)^2 + 4 \cdot 16 = 100$$

$$\sqrt{100} + \frac{12}{2} = 16$$

$$\frac{16}{4} = 4$$

$$x = 4, \quad x^2 = 16$$

If x^2 is to be found first, the following method is applied:

1. If the coefficient of x^2 is greater or less than 1, it is reduced or expanded to 1
2. The square of the coefficient of x is taken and multiplied by C . The result is kept.
3. The square of half the square of the coefficient of x is taken and added to the previously kept result.
4. The square root of this sum is taken.
5. This root is added to C and half the square of the coefficient of x.
6. The resulting value is equal to x^2.

In summary,

$$x^2 = \frac{b^2}{2} + \sqrt{\left(\frac{b^2}{2}\right)^2 + b^2c + c}$$

Example:

b. If $x^2 = 4x + 32$, then what is x, what is x^2?

$4^2 = 16$

$16 \cdot 32 = 512$

$\left(\frac{4^2}{2}\right)^2 + 512 = 64 + 512 = 576$

$\sqrt{576} = 24$

$24 + \frac{4^2}{2} + 32 = 24 + 8 + 32 = 64$

$x^2 = 64, \quad x = 8$

2.3. Higher-Degree Equations

After explaining the solution methods for the six fundamental equation forms, the text states that equations are not limited to these six forms. It then explores how certain higher-degree equations can be solved. The equations discussed in this section are special cases that can be transformed into second-degree equations through the substitution method.

Example:

a. If $x^4 + 3x^2 = 2548$, then what is x^2?

By substituting $x^2 = y$, the given equation is transformed into:

$y^2 + 3y = 2548$

Here, the term $x^2 = y$ is called the intermediate term (*vāsıta*).

To solve the equation, we take half the coefficient of the intermediate term, square it, and add it to the constant term, as the intermediate term is not alone on one side:

$$\left(\frac{3}{2}\right)^2 + 2548 = 2550\frac{1}{4}$$

The square root of this value is taken:

$$\sqrt{2550\frac{1}{4}} = 50\frac{1}{2}$$

Since the intermediate term is on the same side as the highest-degree term, we subtract its half from the square root:

$$50\frac{1}{2} - 1\frac{1}{2} = 49$$

which is y. Thus, we obtain:

$$x^2 = 49$$

b. If $x^4 + 9 = 10x^2$, what is x^2?

By substituting $x^2 = y$, the equation transforms into:

$$y^2 + 9 = 10y$$

To solve this equation, half of the square of the intermediate term's coefficient is taken, and this is subtracted from the number, since the intermediate term is alone on one side of the equation.

$$\left(\frac{10}{2}\right)^2 - 9 = 25 - 9 = 16$$

The square root of this value is taken:

$$\sqrt{16} = 4$$

This root is added to half of the intermediate term's coefficient:

$$4 + \frac{10}{2} = 9$$

The obtained value is equal to x^2, so:

$$x^2 = 9$$

Actually, $x^2 = 1$ is the other solution, but the book does not include it.

c. If $x^4 = 10x^2 + 96$, what is x^2?

By substituting $x^2 = y$, the equation transforms into:

$$y^2 = 10y + 96$$

To solve this equation, half of the square of the intermediate term's coefficient is taken and added to the number, since the intermediate term is not alone on one side of the equation.

$$\left(\frac{10}{2}\right)^2 + 96 = 121$$

The square root of this value is taken:

$$\sqrt{121} = 11$$

Since the intermediate term is on the same side as the lowest degree term, half of it is added to this root:

$$\frac{10}{2} + 11 = 16$$

The obtained value is equal to x^2, so:

$$x^2 = 16$$

d. If $3x^3 + 40 = x^6$, what is x^3?

By substituting $x^3 = y$, the equation transforms into:

$$3y + 40 = y^2$$

To solve this equation, half of the square of the coefficient of x^3 is taken and added to the number.

$$\left(\frac{3}{2}\right)^2 + 40 = 42\frac{1}{4}$$

The square root of this value is taken:

$$\sqrt{42\frac{1}{4}} = 6\frac{1}{2}$$

Since the intermediate term is on the same side as the lowest degree term, half of it is added to this root:

$$\frac{3}{2} + 6\frac{1}{2} = 8$$

The obtained value is equal to x^3, so:

$$x^3 = 8$$

$$x^6 = 64$$

This method can be applied to all equations with similar powers. In a suitable equation, the highest-degree term is considered y^2, and the transformation is applied, yielding a quadratic equation.

CONCLUSION

Khayr al-dīn Khalīl ibn Ibrāhīm, a scholar of the period of Mehmed II the Conqueror, is known for his Persian mathematical works. Among these, *Miftāḥ-i Kunūz-i Erbāb al-Kalam* wa *Misbāh-i Rumūz-i Ashāb al-Raqam* was a foundational work in accounting mathematics, used for a long time in the Ottoman Empire. His work on algebra, *Mushkilkushā-yı Hussāb* wa *Mu'dılnumā-yı Kuttāb*, as its name suggest, was also written for accountants and scribes. However, considering the topics covered, it constitutes an advanced algebra book. From the preface of the work, we learn that Khayr al-dīn Khalīl dedicated his work to Sultan Bayezid II.

This article examines the introduction and the first chapter of *Mushkilkushā-yı Ḥussāb* wa *Mu'ḍılnumā-yı Kuttāb*. The introduction provides the fundamental definitions necessary for algebra. The first chapter explores equation-solving methods. In addition to the six equation forms and classical solutions used to solve quadratic equations, Hayreddin Halil also presents alternative solution methods and explanations for various special cases. Furthermore, in the final section of the first chapter, he explains how some higher-degree equations can be transformed into quadratic equations for solutions.

Khayr al-dīn Khalīl's work differs from classical algebra books both in its approach to topics and its level of complexity. These characteristics suggest that the work was written to educate accountants and scribes in advanced algebra topics. Another important feature of the book is that it is the first work in the Ottoman Empire dedicated solely to algebra.

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OSMANLI ÜLKESİNİN İLK CEBİR KİTABI: HAYREDDİN HALİL B. İBRAHİM'İN MÜŞKİLKUŞÂ-YI HUSSÂB VE MU'ZILNUMÂ-YI KÜTTAB ADLI ESERİNİN İNCELEMESİ

Öz

Hayreddin Halil b. İbrahim, XV. yüzyılda yaşamış ve eserlerini Farsça kaleme almış bir Osmanlı matematikçisidir. Bazı kaynaklar, onun Sultan II. Mehmed'in hocası olan Hoca Hayreddin ile aynı kişi olabileceğini öne sürse de, bu kesin olarak bilinmemektedir. En önemli eserleri Miftâh-i Künûz-i Erbâbi'l-kalem ve Misbâh-i Rumûz-i Ashâbi'r-rakam ile Sultan II. Bayezid'e ithaf ettiği ileri düzey bir cebir kitabı olan Müşkilküşâ-yı Hüssâb ve Mu'zılnumâ-yı Küttâb'tır.

Osmanlı muhasebecileri tarafından kullanılan temel bir aritmetik kitabı olan Miftâh-i Künûz, kesirler, bölünebilirlik kuralları, orantılı sayılar, çarpma, bölme, borç hesaplama ve kök alma gibi konuları ele almaktadır. Daha sonra öğrencisi Pîr Mahmud Sıdkî Edirnevî tarafından Türkçeye çevrilmiştir.

Altı bölümden oluşan bir cebir kitabı olan Müşkilküşâ-yı Hussâb, ise tanımlar, yöntemler ve ileri düzey problem çözme tekniklerini içermektedir. İkinci dereceden, üçüncü dereceden ve daha yüksek dereceden denklemleri ele alarak klasik cebir kitaplarında yaygın olmayan bazı yöntemler sunmaktadır. Ayasofya Koleksiyonu'ndaki el yazması, eserin saray mensupları için kopyalanmış olabileceğini göstermektedir.

Bu çalışma, Müşkilküşâ-yı Hussâb'ın giriş bölümünü ve birinci faslını, Ayasofya ve Şehid Ali Paşa koleksiyonlarındaki yazmaları inceleyerek analiz etmeyi amaçlamaktadır. Araştırma, eserin matematiksel içeriğini, yöntemlerini ve öğretim yaklaşımını ortaya koyarak Osmanlı'da yazılmış bilinen en eski cebir kitabı olarak önemini vurgulamaktadır. Çalışma, eserde sunulan problem çözme tekniklerini inceleyerek Osmanlı muhasebecileri ve kâtipleri için gerekli olan ileri düzey matematik eğitimi hakkında fikir vermeyi hedeflemektedir.

Anahtar Kelimeler: Bilim Tarihi, Hayreddin Halil B. İbrahim, Müşkilküşâ-yı Hüssab, Cebir, Osmanlı Matematik Tarihi